ON THE *GAṆITA-SĀRA-SAṂGRAHA* OF MAHĀVĪRA (*C.* 850 A.D.)

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In the present paper, an attempt has been made to discuss some of the salient features of the precious work, the *Gaṇita-Sāra-Saṃgraha* by the Jaina mathematician Mahāvirācārya (*c.* 850 A.D.). Many topics on algebra and geometry have been discussed in this treatise, and it throws an interesting side-light on the history of Indian mathematics. It enables us to appreciate in a chronological setting, the methodological similarities among the contemporary works on the subject. It has been observed that there is considerable similarity in certain respects among the works of Brahmagupta, Mahāvīrācārya and Bhāskarācārya II. Mahāvīra introduced modifications, improvements or generalisations upon the works of his predecessors particularly Brahmagupta. His work on rational triangles and quadrilaterals deserves special mention as some of the problems discussed by him are not found in the work of any anterior mathematician. Mahāvīra's contributions stimulated the growth of mathematics in India and have a special position in the history of Indian Mathematics.

INTRODUCTION

Mahāvīrācārya (briefly Mahāvīra) was the most celebrated Jain mathematician of the mid-ninth century. He lived within the heart of Deccan (in Mysore area and probably enjoyed the patronage of the Raṣṭrakuṭa king Amoghavarṣa Nṛpatuṅga (814-877 A.D.). It would seem that, geographically, Mahāvīra was far away from the then flourishing mathematical schools in northern and western India ; yet his great work, the *Gaṇita-sāra-saṃgraha* (*GSS*) was an important link in the continuous chain of Indian mathematical texts, which occupied a pride of place, particularly in South India. Its merits were recognized by the enlightened ruler, Raja-Raja Narendra of Rajamahendry who, in the eleventh century A.D., got it translated into Telugu by one Pavuturi Mallana. Through his varied but profound mathematical achievements Mahāvīra occupied a pivotal position between his predecessors (Āryabhaṭa I, Bhāskarācārya I and Brahmagupta) and successors (Srīdhara, Āryabhaṭa II and Bhāskarācārya II). Like other Indian mathematicians, Mahāvīra was not primarily an astronomer although he knew well the science of astronomy and he himself emphasized the importance of mathematics for the study of astronomy.

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The *GSS*, a precious treatise, throws interesting side-light on the history of Indian mathematics and enables us to appreciate, in a chronological setting, the methodological similarities among the contemporary works on the subject.

Strange it would seem though that neither Bhāskarācārya II nor any one of his commentators has made a mention of Mahāvīra's work in their works. Mahāvīra is not even mentioned in Dvidedi's *Gaṇakataraṅgiṇī*. Further, the manuscripts and commentaries relating to *GSS* have come down to us mostly in Telugu and Kannada translations.

The *GSS* consists of nine chapters like the *Bījagaṇita* of Bhāskarācārya II. It deals with operations with numbers except those of addition and subtraction which are taken for granted; squaring and cubing; extraction of square-roots and cube-roots; summation of arithmetic and geometric series; fractions; rule of three; mensuration and algebra including quadratic and indeterminate equations. Twenty-four notational places are mentioned, commencing with the unit's place and ending with the place called *mahā-kṣobha*, and the value of each succeeding place is taken to be ten times the value of the immediately preceding place.

The four fundamental arithmetic operations with zero quantity are discussed in the very first chapter of the text[1]. According to Mahāvīra, "A number multiplied by zero is zero, and that (number) remains unchanged when it is divided by, combined with (or) diminished by zero. Multiplication and other operations in relation to zero give rise to zero; and in the operation of addition, the zero becomes the same as what is added to it." Mahāvīra's statement that a number remains unchanged when divided by zero is obviously not correct. Like Bhāskarācārya*, Mahāvīra should have stated that the quotient in such a case is infinity. But the very mention of operations in relation to zero is enough to show that Mahāvīra was aware of some symbolic representation of the zero quantity. He probably thought that so far as arithmetic is concerned, division by zero is no division at all.. For enumerating the nominal numerals in the first chapter of the book, Mahāvīra mentions certain names so as to denote nine figures from 1 to 9. In the end he gives the names denoting zero and calls all the ten figures by the name of *Saṃkhyā*. Thus zero had a symbol and with the aid of the ten digits and the decimal system of notation numerical quantities of all values could be definitely and accurately expressed.

In the treatment of fractions, Mahāvīra seems to be the first amongst the Indian mathematicians who used the method of least common multiple (L.C.M.) to shorten the process. This he called *niruddha* and he defined it as follows: "The *niruddha* is obtained by means of the continued multiplication of (all) the (possible) common factors of the denominators and (all) their (ultimate) quotients." The process of reducing fractions with equal denominators is thus described by him

*Bhāskarācārya calls the quotient of such zero divisions *Khahara* and rightly assigns to it the value of infinity.

as : "the new numerators and denominators, obtained as products of multiplication of (each original) numerator and denominator by the (quotient of the) *niruddha* divided by the denominator give fractions with the same denominator."

The works of Brahmagupta, Mahāvīra and Bhāskara are similar in spirit but entirely different in detail. For example, each treats the area of a segment of a circle and of polygons, but gives different results. It may be mentioned here that the formula for the area of a segment of a circle as given in the *GSS*[3], and *Laghu Kṣetra-Samāsa*[4] of Ratneśvara Suri (15th cent A.D.) 1 viz.

$$\text{chord} \times \frac{\text{height}}{4} \times \sqrt{10}$$

is not correct. This formula has probably been obtained by analogy of the rule for the area of a semi-circle, which area is evidently

$$\text{diameter} \times \frac{\text{radius}}{4} \times \pi.$$

The so called *janya* operation[5], mentioned in the *GSS* in calculations relating to the measurement of areas, is akin to the work found in Brahmagupta,(but none of the problems is the same.

Both Brahmagupta and Mahāvīra give the formula for the area of a quadrilateral in terms of the sides a, b, c, d and s (where $2s = a+b+c+d$) as

$$\sqrt{(s-a)\ (s-b)\ (s-c)(s-d)}$$

and neither of them has observed that this formula holds good only for a cyclic figure.

For the volume of a sphere, Mahāvīra gave an approximate formula as $\frac{9}{2}\ (\frac{1}{2}d)^3$and the accurate one as $\frac{9}{10} \cdot \frac{9}{2}\ (\frac{1}{2}d)^3$. The latter gives the value of π as 3.0375. Mahāvīra also treated *kuṭṭaka*[6] (simple and simultaneous indeterminate equations of the first degree, viz. $\frac{ax \pm c}{b} = y$). What is more, he treated in an ingenious manner the ellipse, for the area of which he gave the following formula:

$$(\text{circumference}) \times \frac{1}{4}(\text{semi-minor axis}).$$

But it may be observed that this formula is not correct. In fact Mahāvīra might have given this formula on the analogy of the area of a circle in the form $\pi d \times \frac{1}{4}d$, where d is the diameter.

In keeping with the tradition of those days, many topics on algebra and geometry have been discussed in the *GSS*. What follows is a brief account of some of them.

1. Solution of Quadratic Equations

Mahāvīra knew that a quadratic equation has two roots. We shall substantiate this by taking two problems given in his work and the rules given therein for solving such an equation.

Problems involving the square of the unknown :

Example 1. One-sixteenth part of a collection of peacocks as multiplied by itself (i.e. by the same 1/16 part of the collection), was found on a mango tree; 1/9 of the remainder as multiplied by itself, as also (the remaining) fourteen (peacocks) were found in a grove of *tamāla* trees. How many are they (in all) ?[7]

Solution : If x be the total number of peacocks, the problem leads to the quadratic equation

$$\frac{x}{16}\cdot\frac{x}{16}+\left(\frac{1}{9}\cdot\frac{15}{16}x\right)\left(\frac{1}{9}\cdot\frac{15}{16}x\right)+14=x.$$

This is an equation of the form

$$\frac{a}{b}x^2-x+c=0.$$

Mahāvīra[8] has given the following rule for solving it :

"From the (simplified) denominator (of the specified compound fractional part of the unknown collective quantity), divided by its own (related) numerator (also simplified), subtract four times the given known part (of the quantity), then multiply this (resulting difference) by that same (simplified) denominator (dealt with as above). The square-root (of this product) is to be added to as well as sub tracted from that (same) denominator (so dealt with); (then) the half (of either) of these (two quantities resulting as sum or difference) is the unknown collective quantity (required to be found out.)"

Applying this rule, the above equation gives

$$x=\frac{\frac{b}{a}\pm\sqrt{\left(\frac{b}{a}-4c\right)\frac{b}{a}}}{2}$$

Example 2. One-twelfth part of a pillar, as multiplied by 1/30 part thereof was to be found under water; 1/20 of the remainder, as multiplied by 3/16 thereof, was found (buried) in the mire (below); and 20 *hastas* of the pillar were found in the air (above the water). O friend, you give out the measure of the length of the pillar.[9]

Solution. If x be the height of the pillar, the problem leads to the equation

$$\left(x-\frac{x^2}{12.30}\right)-\frac{1}{20}\cdot\frac{3}{16}\left(x-\frac{x^2}{12.30}\right)^2=20.$$

Mahāvīra put $\left(x - \frac{x^2}{360}\right) = y$, and then solved the quadratic equation

$$y - \frac{3}{320}y^2 = 20.$$

The equation thus gives four values of x. Mahāvīra considered only the rational solutions 240 and 120 and discarded the irrational numbers.

Certain other problems[10] given in *GSS* lead to equations of the type.

$$\left(\frac{a}{b}x \mp d\right)^2 + c = x.$$

Mahāvīra gave the following rule for solving such equations.

"(Take) the half of the denominator (of the specified fractional part of the unknown collective quantity), as divided by its own (related) numerator, and as increased or diminished by the (given) known quantity which is subtracted from or added to (the specified fractional part of the unknown collective quantity). The square-root of the square of this (resulting quantity), as diminished by the square of (the above known) quantity to be subtracted or to be added and (also) by the known remainder (of the collective quantity), when added to or subtracted from the square-root (of the square quantity mentioned above) and then divided by the (specified) fractional part (of the unknown collective quantity), gives the(required) value (of the unknown collective quantity)".[11]

According to the rule, the above equation gives

$$x = \left[\pm\sqrt{\left(\frac{b}{2a} \pm d\right)^2 - d^2 - c} + \left(\frac{b}{2a} \pm d\right)\right] \div \frac{a}{b}.$$

2. Minor Methods of Cubing

Mahāvīra has given the following rules for the operation of a given quantity :

Rule 1 : "The product obtained by the multiplication of any (given) quantity by that (given quantity) as diminished by a chosen quantity, and (then again) by that (given quantity) as increased by the (same) chosen quantity, when combined with the square of the chosen quantity as multiplied by the least (of the above three quantities) and (combined) also with the cube of the chosen quantity, gives rise to a cubic quantity."

Symbolically expressed, this rule means

$$a(a+b)\,(a-b) + b^2\,(a-b) + b^3 = a^3$$

Rule 2. "The summing up of a series in arithmetical progression (A.P.), of which the first term is the quantity (the cube whereof is) required, the common difference is twice this quantity, and the number of terms is (equal to) this (same given) quantity, gives rise to the cube of the given quantity.

Or, the square of the quantity (the cube whereof is required), when combined with the product (obtained by the multiplication) of this given quantity diminished by *one* by the sum of a series in A.P. in which the first term is *one*, the common difference is *two* and the number of terms is (equal to) the given quantity, gives rise to the cube of the given quantity."[13]

Algebraically, this rule means

(*i*) $a^3 = a+3a+5a+7a+\ldots\ldots$ to a terms

(*ii*) $a^3 = a^2+(a-1)\ (1+3+5+7+\ldots\ldots$ to a terms)

Rule 3. "In an arithmetically progressive series, wherein one is the first term as well as the common difference, and the number of terms is (equal to) the given number, multiply the preceding terms by the immediately following ones. The sum of the products(so obtained), when multiplied by three and combined with the last term(in the above series in A.P.) becomes the cube (of the given quantity."[14]

Algebraically, this rule means

$$3[1.2+2.3+3.4+4.5+\ldots.+(a-1)a]+a = a^3$$

$$\text{or} \quad 3\left[\sum_{r=2}^{a} r(r-1)\right]+a = a^3.$$

Rule 4. "(In a given quantity), the squares of (the number represented by the figures in) the last place as also (by those in) the other (remaining places) are taken ; and each of these (squares) is multiplied by the number of the other place and also by *three* ; the sum of the two (quantities resulting thus), when combined again with the cubes of the numbers corresponding to all the (optional) places (gives rise to) the cube (of the given quantity.)"[15]

Symbolically, this rule means

$$3a^2b+3ab^2+a^3+b^3 = (a+b)^3$$

To make the rule general and applicable to numbers having more than two places it is clearly implied here that

$$3a^2(b+c)+3a(b+c)^3+a^3+(b+c)^3=(a+b+c)^3$$

It may be noted here that the foregoing formulae, as given by Mahāvīra, are well known in present day algebra.

3. Treatment of Geometric Series

Mahāvīra called the process of summation of series from which the first few terms are omitted as *Vyutkalita*,[16] and has given all the formulae for geometric progression (G.P.) thus earning for himself a prominent position in this respect. For the sum of n terms of a G.P., he gave the formulae

$$\frac{a\ r^{n-1}.\ r-a}{(r-1)} \text{ or } \frac{a\ (r^n-1)}{(r-1)},$$

where a is the first term and r the common ratio of the given series.

In connection with the treatment of geometric series, Mahāvīra has discussed the methods of solving the following types of equations :

$$(i)\ a\,r^n = q$$

$$(ii)\ a\left(\frac{r^n-1}{r-1}\right) = p,$$

where q is the *gunadhana* or $(n+1)$th term of a geometric series and p is its sum.

Mahāvīra's rule for finding out the common ratio r from the equation (i), in relation to a given *gunadhana*, is :

"The *gunadhana* when divided by the first term becomes equal to the (self-multiplied) product of a certain quantity in which (product) that (quantity) occurs as often as the number of terms (in the series) ; and this (quantity) is the (required) common ratio."[17]

$$\text{i.e. } r = \sqrt[n]{\frac{q}{a}}\ .$$

In other words, r is the nth root of $\frac{q}{a}$. For finding out the common ratio r from the equation (ii), in relation to the (given) sum of a series in G.P., Mahāvīra says :

"That (quantity) by which the sum of the series divided by the first term and (then) lessened by one is divisible throughout (when this process of division after the subtraction of one is carried on in relation to all the successive quotients) time after time—(that quantity) is the common ratio."[18]

To illustrate the above rule, we consider the following illustration from the *GSS.* :

Example. "When the first term is 3, the number of terms is 6, and the sum is 4095 (in relation to a series in G.P.). When is the value of the common ratio ?"[19]

Sd. Here $3\ \dfrac{(r^6-1)}{(r-1)} = 4095$

Dividing 4095 by 3, we get 1365.

Now, $1365-1 = 1364$.

Choosing by trial a divisor 4, we have

$$\frac{1364}{4} = 341\ ;\ 341-1 = 340\ ;\ \frac{340}{4} = 85\ ;\ 85-1 = 84\ ;$$

$$\frac{84}{4} = 21\ ;\ 21-1 = 20\ ;\ \frac{20}{4} = 5\ ;\ 5-1 = 4\ ;$$

$\frac{4}{4} = 1.$ Hence 4 is the common ratio.

The principle underlying this method is :

$$a\left(\frac{r^n-1}{r-1}\right)\div a=\frac{r^n-1}{r-1}\ ;\ \text{and}\ \left(\frac{r^n-1}{r-1}\right)-1=\frac{r^n-r}{r-1},$$

which is obviously divisible by r.

4. Unit Fractions

Mahāvīra has given a number of interesting as well as accurate rules for expressing a given fraction as the sum of a number of unit fractions. The ancient Egyptian mathematician Ahmes (second century B.C.) gave a table for $n = 1$ to 49 expressing the fraction $\frac{2}{2n+1}$ as a sum of unit fractions. But these methods were empirical. Mahāvīra's rules are, to be sure, not empirical methods. We shall quote here a few of his methods as given in the *GSS*.

(*i*) To express 1 as the sum of a number (n) of unit fractions. Mahāvīra's rule for this is :[20]

"When the sum of the (different fractional) quantities is one and their numerator is 1, the (required) denominators are such as, beginning with one, are in order multiplied (successively) by three, the first and the last (denominators so obtained) being (however) multiplied (again) by 2 and 2/3 (respectively)".

Algebraically, this rule means

$$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots\dots+\frac{1}{3^{n-2}}+\frac{1}{2.3^{n-2}}$$

For $n = 5$, this gives

$$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{54}.$$

(*ii*) To express 1 as the sum of $(2n-1)$ (i.e. an odd number) of unit fractions, Mahāvīra's rule is :[21]

"When the sum of the (different fractional) quantities, having *one* for each of their numerators, is *one*, the (required) denominators are such as, beginning with *two*, go on (successively) rising in value by *one*, each (such denominator) being (further) multiplied by that (number) which is (immediately) next to it (in value) and then halved."

Algebraically, this means

$$1=\frac{1}{2.3.\frac{1}{2}}+\frac{1}{3.4.\frac{1}{2}}+\dots\dots+\frac{1}{(2n-1).2n.\frac{1}{2}}+\frac{1}{2n.\frac{1}{2}}.$$

For $n = 4$,

$$1=\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}+\frac{1}{4}\ .$$

(*iii*) To express a given unit fraction as the sum of r fractions with given numerators $a_1, a_2, \ldots\ldots, a_r$, his rule is :[22]

"When the sum (of certain intended fractions) has *one* for its numerator, then (their required denominators are arrived at by taking) the denominator of the sum to be that of the first (fraction), and (by taking) this (denominator) combined with its own (related) numerator to be (the denominator) of the next (fraction) and so on, and then by multiplying (further each such denominator in order) by that which is (immediately) next to it, the last (denominator) being (however multiplied) by its own (related) numerator."

Algebraically this rule means

$$\frac{1}{n}=\frac{a_1}{n(n+a_1)}+\frac{a_2}{(n+a_1)(n+a_1+a_2)}+\ldots+\frac{a_{r-1}}{(n+a_1+a_2+\ldots+a_{r-2})(n+a_1+a_2+\ldots a_{r-1})}$$

$$+\frac{a_r}{a_r(n+a_1+a_2+\ldots+a_{r-1})}.$$

By taking $a_1=a_2=\ldots=a_r=1$, we express $1/n$ as the sum of r unit fractions. When the a's are not unity, the fractions may not be in their lowest terms.

For $n=5$, $r=4$;

$$\frac{1}{5}=\frac{1}{5.6}+\frac{1}{6.7}+\frac{1}{7.8}+\frac{1}{8}.$$

(*iv*) To express any fraction as the sum of unit fractions, his rule is :[23]

"The denominator (of the given fraction), when combined with an optionally chosen number and then divided by the numerator so as to leave no remainder, becomes the denominator of the first numerator (which is one); the optionally chosen quantity when divided by this and by the denominator of the sum is the remainder. To this remainder the same process is applied."

If $\frac{p}{q}$ $(p<q)$ is the given fraction, choose the number i so that $\frac{q+i}{p}$ is an integer, say r; then the above rule gives :

$$\frac{p}{q}=\frac{1}{r}+\frac{i}{r.q}$$

The first term is a unit fraction and a similar process can be used for the remainder $\frac{i}{r.q}$ to get other unit fractions.

(*v*) To express a unit fraction as the sum of two other unit fractions, Mahāvīra states the rule as follows :[24]

"The denominator of the (given) sum multiplied by a properly chosen number is the (first) denominator; and this (denominator) divided by the (previously) chosen (number) as lessened by one gives rise to the other (required denominator)" or, "When in relation to the denominator of the (given) sum (any chosen) divisor (thereof) and the quotient (obtained therewith) are (each) multiplied by their sum, they give rise to the two (required) denominators."

Expressed algebraically, these rules give

$$\frac{1}{n}=\frac{1}{p.n}+\frac{1}{\frac{p.n}{n-1}},$$

where the integer p is so chosen that n is divisible by $(p-1)$; and

$$\frac{1}{a\,b}=\frac{1}{a(a+b)}+\frac{1}{b(a+b)},\ (b \geqslant 1).$$

(*vi*) To express any fraction as the sum of two other fractions whose numerators are given, the rule of Mahāvīra is :[25]

"(Either) numerator multiplied by a chosen (number), then combined with the other numerator, then divided by the numerator of the (given) sum so as to leave no remainder, and then divided by the (above) chosen number and multiplied by the denominator of the (above) sum, gives rise to a (required) denominator The denominator of the other (fraction), however, is this (denominator) multiplied by the (above) chosen (quantity)."

Algebraically, this rule means :

$$\frac{m}{n}=\frac{a}{\left(\frac{ap+b}{m}\right)\cdot\frac{n}{p}}+\frac{b}{\left(\frac{ap+b}{m}\right)\cdot\frac{n}{p}\,p}.$$

Here the chosen number p must be a divisor of n, and be such that $(ap+b)$ is divisible by m.

If we take $p = n$, and if $(an+b)$ is divisible by m, then

$$\frac{m}{n}=\frac{a}{\left(\frac{an+b}{m}\right)}+\frac{b}{\left(\frac{an+b}{m}\right)n}.$$

(*vii*) To express a given fraction as the sum of an even number of fractions with given numerators, Mahāvīra's rule for this is :[26]

"The numerator (of one of the fractions) as multiplied by the denominator of the sum, when combined with the other numerator and then divided by the numerator of the sum, gives rise to the denominator of one (of the fractions). This (denominator), when multiplied by the denominator of the sum, becomes the denominator of the other (fraction)."

It may be observed that this rule is only a particular case of the previous rule as the denominator of the sum is itself substituted in this rule for the quantity to be chosen in the previous rule.

5. Rational Right Triangles

The Indian mathematicians attached great importance to the solution of rational triangles and quadrilaterals,[27] this was increasingly evident after the sixth century A.D.But Mahāvīra's work differs from those of the others in respect of the basic definitions he expounded concerning various figures (triangles, quadrilaterals etc.).

According to Mahāvīra a triangle or a quadrilateral whose sides, altitude and other dimensions can be expressed in terms of rational numbers is called *janya* (meaning that which is generated or formed). No wonder then that the section dealing with the rational triangles and quadrilaterals has been given the sub-title *janya-vyāvahāra* (*janya* operation). Numbers that are employed in forming a particular figure are called *bīja-saṃkhyā* or simply *bīja* (element) and two such are invariably given for the derivation of trilateral and quadrilateral figures dependent on them.

Mahāvīra's rule for arriving at a longish quadrilateral figure with optionally chosen numbers as *bījas* is :

"In the case of the optionally derived longish quadrilateral figure the diffeence between the squares (of the *bīja* numbers) constitutes the measure of the perpendicular-side, the product (of the *bīja* numbers) multiplied by *two* becomes the (other) side, and the sum of the squares (of the *bīja* numbers) becomes the hypotenuse."[28]

Thus if a and b are the *bīja* numbers, then a^2-b^2 is the measure of the perpendicular, $2ab$ that of the other side, and a^2+b^2 that of the hypotenuseof anoblong. Thus it is evident that the *bījas* are numbers with the aid of the product and the squares whereof, as forming the measure of the sides, a right-angled triangle may be constructed.

In *GSS* we come across with statements such as :

"Form the generated figure from the *bija* 2, 3."[29]

"Form another from half the base of the figure (rectangle) from the *bīja* 2, 3,[30] etc.

Thus, according to Mahāvīra, forming a rectangle from the *bīja* a, b means taking a rectangle with the perpendicular-side, base and diagonal as a^2-b^2, $2ab$, and a^2+b^2 respectively. Mahāvīra's statement in this regard resembles that of Diophontus[31] (second century A.D.). What Diophontus calls "forming a right angled triangle from a, b", Mahāvīra calls" forming a longish quadrilateral or rectangle from a, b". He never speaks of 'right angled triangle'. However, he recognised only three kinds of triangles—equilateral, isosceles and scalene.[32]

5.1.[1] *Right triangles having a given hypotenuse* (*C*)

Mahāvīra has given three rules for the solution of the equation $x^2+y^2 = c^2$, in rational integers.

Rule 1. "The square-roots of half the sum and of half the difference of the measure of the hypotenuse and the square of a (suitably) chosen optional number are the *bījas* (the elements)"[33]

In other words, the required solution will be obtained from the *bījas* $\sqrt{\frac{1}{2}(c+p^3)}$, $\sqrt{\frac{1}{2}(c-p^2)}$, where p is any rational number ; and then the solution is p^2, $\sqrt{c^2-p^4}$, c.

Rule 2. "The square-root of the difference between the squares of the hypotenuse and of a (suitably chosen) optional number forms, along with that chosen number, the perpendicular-side and the other side respectively."[34]

According to this rule, the solution is

$$p, \sqrt{c^2-p^2}, c$$

It may be observed that in the aforesaid two solutions $\sqrt{c^2-p^4}$ or $\sqrt{c^2-p^4}$ may not be rational unless p is suitably chosen.

We now state Mahāvīra's third rule, which is of greater importance :

Rule 3. "Each of the various figures (rectangles) that are derived with the aid of the given (*bījas*) is written down ; and by means (of the measure) of its diagonal the (measure of the) given diagonal is divided. The perpendicular-side, the base, and the diagonal (of this figure) as multiplied by the quotient (here) obtained, give rise to the perpendicular-side, the base and the diagonal (of the required figure)".[35]

This rule is based on the principle that the sides of a right-angled triangle vary as the hypotenuse, although for the same measure of the hypotenuse there may be different sets of values for the sides.

For the general solution of the rational right triangle, viz. m^2-n^2, $2mn$, m^2+n^2 Mahāvīra's third rule reduces it to the ratio $\frac{c}{m^2+n^2}$, so that all rational right triangles having a given hypotenuse c will be given by

$$\left(\frac{m^2-n^2}{m^2+n^2}\right)c, \quad \left(\frac{2mn}{m^2+n^2}\right)c, \; c,$$

As an illustration, Mahāvīra found four rectangles 39, 52 ; 25, 60 ; 33, 56 ; and 16, 63 all having the same diagonal (in value) 65.[36]

A fact of historical significance is that this method was rediscovered in Europe by Leonardo Fibonacci of Pisa (thirteenth century A.D.) and Vieta (sixteenth century A.D.).

5.2. *Problems involving areas and sides*

Mahāvīra has treated rational rectangles (or squares) in which the area will be numerically multiple or submultiple of a side, diagonal or perimeter, or in general a linear combination of the sides and the diagonal. If x and y are the sides, and z the diagonal, the problem relates to the solution of the equations

$$\left.\begin{array}{r} x^2+y^2 = z^2, \\ mx+ny+pz = rxy, \end{array}\right\} \quad (i)$$

where m, n, p, r $(r \neq o)$ are any rational numbers.

The method adopted for solution is the same as the third one in the previous case. Starting with any rational solution of

$$x'^2+y'^2 = z'^2,$$

Mahāvīra says ; calculate the value of $mx'+ny'+pz' = Q$, say.

The required solutions of (1) will be obtained by reducing the values of x', y', z' in the ratio $\frac{Q}{rx'y'}$,

$$\left.\begin{array}{l} \text{Thus} \quad x = \dfrac{x'Q}{rx'y'} = \dfrac{Q}{ry'}, \\ \qquad\quad y = \dfrac{y'Q}{rx'y'} = \dfrac{Q}{rx'}, \\ \qquad\quad z = \dfrac{z'Q}{rx'y'}. \end{array}\right\}$$

It is readily verified that these values of x, y, z satisfy the equations (i).

Mahāvīra has given several illustrations, some of which are very interestng :

Example 1. "In a rectangle, the area is (numerically) equal to the perimeter ; in another rectangle the (numerical) measure of the area is equal to that of the diagonal. What is the measure of the base (in each of these cases)"?[38]

Example 2. "In the case of a longish quadrilateral figure, (the numerical measure of) twice the diagonal, three times the base and four times the perpendicular-side being taken, the measure of the perimeter is added to them. Twice (this sum) is the (numerical) measure of the area. Find out the measure of the base".[39]

5.3. *Problems involving sides but not areas*

Mahāvīra obtained right triangles the sum of whose sides when multiplied by arbitrary rational numbers has a given value. Expressed symbolically, this requires solving the equations

$$\left.\begin{array}{r} x^2+y^2 = z^2, \\ rx+sy+tz = A, \end{array}\right\} \quad (1)$$

where r, s, t and A are known rational numbers.

Mahāvīra's method of solution is the same as described above. Starting with the general solution of

$$x'^2+y'^2 = z'^2,$$

we have to calculate the value of

$$rx'+sy'+tz' = P, \text{ say,}$$

According to Mahāvīra, the solution of (1) will be

$$x = \frac{x' A}{P}, \; y = \frac{y' A}{P}, \; z = \frac{Z'A}{P}.$$

In particular, if $r = S = 0$, $t = 1$, $A = C$ then $P = z'$. Hence with the general solution m^3-n^2, $2mn$, m^2+n^2 of $x'^2+y'^2= z'^2$, we obtain the already known solution

$$\left(\frac{m^2-n^2}{m^2+n^2}\right) c, \; \left(\frac{2mn}{m^2+n^2}\right) c, \; c$$

of rational right triangles having a given hypotenuse (c).

Similarly if $s=t = 0$, $r = 1$, $A = a$, we obtain the solution

$$a, \; \left(\frac{2mn}{m^2-n^2}\right) a, \; \left(\frac{m^2+n^2}{m^2-n^2}\right) a$$

of rational right triangles having a given leg (a).

Again if $r = s = 2$, $t = 0$, $A = 1$ then $p = 2$ $(m -n +2mn)$, so that all rectangles having the same perimeter unity will be

$$\frac{m^2-n^2}{2(m^2-n^2+2mn)}, \quad \frac{2mn}{2(m^2-n^2+2mn)}$$

where m, n, are any rational numbers.

The isoperimetric right triangles will be

$$\left(\frac{m-n}{2m}\right) s, \; \left(\frac{n}{m+n}\right) s, \; \left(\frac{m^2+n^2}{2m(m+n)}\right) s,$$

where s is the given perimeter.

We now take two illustrative problems from the *GSS*.

1. "In the case of a longish quadrilateral figure, the (numerical) measure of the perimeter is 1. Tell me quickly, after calculating, the measure of the perpendicular-side and that of the base".[40]

2. "(Find) a rectangle in which the (numerical) measure of twice the diagonal, three times the base, and four times the perpendicular, on being added to the (numerical) measure of the perimeter, become equal to unity."[41]

5.4. *Pairs of Rectangles*

Mahāvīra discussed three types of triangles as follows :

(*i*) Whose perimeters are equal, but the area of one is double that of the other, or (*ii*) whose areas are equal, but the perimeter of one is double that of the other, or

(*iii*) the perimeter of one is double that of the other, and the area of the latter is double that of the former.

These are the particular cases of the following general problem adumbrated in Mahāvīra's rule :

If (x, y) and (u, v) represent the base and the perpendicular-side respectively of two rectangles, these problems mean the solution of the equations.

$$\left.\begin{aligned} m(x+y) &= n(u+v), \\ pxy &= quv, \end{aligned}\right\}$$

where m, n, p, q are known integers.

Mahāvīra's method[42] leads to two solutions of the general problem, but the problem is really indeterminate.

Mahāvīra's work on 'rational triangles and quadrilaterals contains many other problems of similar nature, and a number of illustrative examples are given in the *GSS*. But it is noteworthy that his investigations in this particular field[43] have certain remarkable features, and they deserve a special consideration for the following two reasons :

(*i*) He treated certain problems, on rational triangles and quadrilaterals, which are not found in the work of any anterior mathematician, e.g. problems on right triangles involving areas and sides,rational triangles and quadrilaterals having a given area or circum-diameter, pairs of isosceles triangles etc ; (*ii*) in the treatment of other common problems, Mahāvīra has introduced modifications, improvements or generalisations upon the works of his predecessors,particularly of Brahmagupta (sixth century A.D.)

It may be remarked here that the credit, which Mahāvīra rightly deserves for his discovery of certain methods for the solution of rational triangles and quadrilaterals has gone almost unnoticed by historians of ancient mathematics, like L.E. Dickson.[44] A reassessment of Mahāvīra's position in this field is indeed necessary.

Mahāvīra, by his protracted achievements in several branches of mathematics, has a distinct position specially in the history of Indian mathematics. His contributions stimulated the growth of mathematics, particularly in the field of rational right triangles and quadrilaterals.

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